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Semihyperbolic Mappings

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Summary. Semihyperbolic dynamical systems generated by Lipschitz mappings are investigated. A special form of robustness of topological entropy under perturbations of a semihyperbolic mapping is discussed, and weakened forms of persistence and of structural stability are considered. Proofs are based on the concept of bi-shadowing, which is a stronger version of the shadowing lemma.

Key words. Hyperbolic systems, semihyperbolicity, topological entropy, persistence, structural stability

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1. Introduction

Many useful properties of hyperbolic diffeomorphisms are retained by the semihyperbolic mappings that were introduced for local diffeomorphisms in [5] and extended to Lipschitz mappings in [6] and [7]. See also Anosov [1] and Ombach [13], where related concepts are discussed. Walters [16] discusses similar properties for expansive shadowing homeomorphisms and there is a compilation of results in this direction in Aoki and Hiraide [2].

That semihyperbolic mappings, which are not necessarily invertible, also share many of these properties was shown in previous papers for expansivity [6] and shadowing [7]. In this paper the investigation is extended to include topological entropy and a weakened form of structural stability. The use of expansivity and

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shadowing in this paper is principally similar to that in [2] and [16], but in the wider context of semihyperbolicity.

Definitions of semihyperbolicity and of shadowing and expansivity are recalled in Sections 2 and 3, respectively, along with statements of results from [6] and [7]. A result asserting a special form of robustness of topological entropy under perturbations of a semihyperbolic mapping is discussed in Section 4. The weakened form of structural stability—essentially the structural or topological stability of related shift operators—is discussed in Section 5. Qualitative results similar to those of the present paper can also be established in other contexts, for example, that of pre-hyperbolicity (Ruelle [14]) or of nonsmooth hyperbolic homeomorphisms (Mane [10], Ombach [13]) using the principal tool of the paper, the concept of bi-shadowing. An advantage of the present approach is that simple explicit estimates can be combined with the study of nonsmooth Lipschitz mappings which need not be invertible. This last feature is especially important, for instance, in the analysis of control systems or of systems with hysteresis ([4], [8]) and in a theoretical investigation of computer realizations of dynamical systems with chaotic behaviour.

2. Semihyperbolic Systems in $\mathbf{P}^d$

Let $|\cdot|$ be a fixed norm on $\mathbb{R}^d$ and let $\mathfrak{X}$ be an open bounded subset of $\mathbb{R}^d$. Denote by $C = C(\mathfrak{X}, \mathbb{R}^d)$ the space of continuous bounded mappings $g: \mathfrak{X} \mapsto \mathbb{R}^d$ with the norm $\|g\|_C = \sup_{x \in \mathfrak{X}} |g(x)|$ and by $\mathscr{L} = \mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$ the space of Lipschitz mappings $f: \mathfrak{X} \mapsto \mathbb{R}^d$ with the norm $\|f\|_{\mathscr{L}} = \|f\|_C + \mathrm{Lip}(f)$, where

$$\mathrm{Lip}(f) = \inf\{\alpha : |f(x) - f(y)| \le \alpha |x - y|,\ x, y \in \mathfrak{X}\}.$$

A four-tuple $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ of nonnegative real numbers is called a *split* if

$$\lambda_s < 1 < \lambda_u, \qquad (1 - \lambda_s)(\lambda_u - 1) > \mu_s \mu_u.$$

Let K be a compact subset of $\mathfrak{X}$ and let $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ be a split. A mapping $f \in \mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$ is said to be $\mathbf{s}$*-semihyperbolic on the set* K if there exist positive real numbers k, δ such that for each $x \in K$ there exists a splitting (decomposition)

$$\mathbb{R}^d = E_x^s \oplus E_x^u \tag{2.1}$$

with corresponding projectors P_x^s and P_x^u satisfying the following three properties:

SH0. $\dim(E_x^s) = \dim(E_{f(x)}^s)$ if $x, f(x) \in K$.

SH1. $\sup_{x \in K}\{|P_x^s|, |P_x^u|\} \le k$.

SH2. The inclusion $x + u + v \in \mathfrak{X}$ and the inequalities

$$|P_{f(x)}^s(f(x + u + v) - f(x + \tilde{u} + v))| \le \lambda_s |u - \tilde{u}|,$$
$$|P_{f(x)}^s(f(x + u + v) - f(x + u + \tilde{v}))| \le \mu_s |v - \tilde{v}|,$$
$$|P_{f(x)}^u(f(x + u + v) - f(x + \tilde{u} + v))| \le \mu_u |u - \tilde{u}|,$$
$$|P_{f(x)}^u(f(x + u + v) - f(x + u + \tilde{v}))| \ge \lambda_u |v - \tilde{v}|,$$

hold for all $x \in K$ with $f(x) \in K$ and all $u, \tilde{u} \in E_x^s$, $v, \tilde{v} \in E_x^u$ such that $|u|, |\tilde{u}|, |v|, |\tilde{v}| \le \delta$.

The first three inequalities in SH2 are just local Lipschitz conditions on the projections of the mapping f while the last one is an expansivity condition which implies a local invertibility in the unstable direction of f. Note that continuity in x of the splitting subspaces E_x^s, E_x^u or of the projectors P_x^s, P_x^u is not assumed here nor is invariance of the splitting subspaces, as is the case in the definition of hyperbolicity of a diffeomorphism. The concept of semihyperbolicity is, in a sense, similar to, but distinct from that of prehyperbolic mappings in [14].

The map f is said to be *continuously semihyperbolic on* K if it is s-semihyperbolic on K and, moreover, the splitting (2.1) is continuous in x. A subset $\mathscr{F}$ of $\mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$ is called *uniformly semihyperbolic on a compact subset* K if there exist positive numbers k, δ and a split $\mathbf{s}$ such that each mapping $f \in \mathscr{F}$ is s-semihyperbolic on K with the same constants k, δ. In the following lemma $\mathscr{O}_\varepsilon(K)$ denotes the open ε-neighbourhood of the subset K.

Lemma 1. *Suppose that the mapping* $f \in \mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$ *is continuously semihyperbolic on a compact subset* $K \subset \mathfrak{X}$. *Then there exists an* $\varepsilon > 0$ *such that* $\overline{\mathscr{O}_\varepsilon(K)} \subset \mathfrak{X}$ *and the set of mappings*

$$\mathscr{F} = \{g \in \mathscr{L}(\mathfrak{X}, \mathbb{R}^d): \|g - f\|_{\mathscr{L}} \le \varepsilon\} \tag{2.2}$$

is uniformly semihyperbolic on $\overline{\mathscr{O}_\varepsilon(K)} \subset \mathfrak{X}$.

Proof. Let (2.1) be a continuous decomposition of f such that properties SH0–SH2 hold for f with the split $\mathbf{s}^*$ and positive constants k^* and δ^*, L^*. For each $x \in \mathfrak{X}$ let $\pi_K(x)$ be any one of nearest points in K to x and consider the decomposition

$$\mathbb{R}^d = E^s_{\pi(x)} \oplus E^u_{\pi(x)}, \qquad x \in \mathfrak{X}. \tag{2.3}$$

Choose constants $k > k^*$, $\delta < \delta^*$, $L > L^*$ and some split $\mathbf{s} = \{\lambda_s, \lambda_u, \mu_s, \mu_u\}$ satisfying $\lambda_s > \lambda_s^*$, $\mu_s > \mu_s^*$, $\lambda_u < \lambda_u^*$, $\mu_u < \mu_u^*$. Then there exists a small $\varepsilon > 0$ such that each mapping in the set (2.2) satisfies properties SH0–SH2 on the compact set $\overline{\mathscr{O}_\varepsilon(K)} \subset \mathfrak{X}$ with the decomposition (2.3), split $\mathbf{s}$ and constants k, δ, L just introduced. □

3. Bi-shadowing and Expansivity

A *trajectory* of a discrete-time dynamical system on the state space $\mathfrak{X}$ generated by the mapping $f: \mathfrak{X} \mapsto \mathfrak{X}$ is a sequence $\mathbf{x} = \{x_n\} \subset \mathfrak{X}$ satisfying $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots, n_+$ or $n = -n_-, \ldots, -1, 0, 1, \ldots, n_+$, where $n_\pm \le \infty$. A sequence $\mathbf{y} = \{y_n\} \subset \mathfrak{X}$ with

$$|y_{n+1} - f(y_n)| \le \gamma, \tag{3.1}$$

for such n and some $\gamma > 0$, is a γ-pseudotrajectory of the dynamical system. In both cases the qualifier "finite" may be appended when $n_\pm < \infty$ and "infinite" otherwise.

Let $\mathbf{Tr}(f, \mathfrak{Y}, \gamma)$ denote the totality of finite or infinite γ-pseudotrajectories belonging entirely to a subset $\mathfrak{Y} \subseteq \mathfrak{X}$. Since a trajectory can be regarded as a 0-pseudotrajectory, the set of all finite or infinite trajectories which belong entirely to $\mathfrak{Y}$ will be denoted by $\mathbf{Tr}(f, \mathfrak{Y}, 0)$. In fact, a trajectory is also a γ-pseudotrajectory for any $\gamma > 0$, so $\mathbf{Tr}(f, \mathfrak{Y}, 0) \subset \mathbf{Tr}(f, \mathfrak{Y}, \gamma)$ with strict inclusion as there are obviously γ-pseudotrajectories which are not trajectories.

A dynamical system generated by a mapping $f\colon \mathfrak{X} \mapsto \mathfrak{X}$ is said to be *bi-shadowing* with positive parameters α and β on a subset $\mathfrak{Y}$ of $\mathfrak{X}$ if for any given finite pseudotrajectory $\mathbf{y} = \{y_n\} \in \mathbf{Tr}(f, \mathfrak{Y}, \gamma)$ with $0 \leq \gamma \leq \beta$ and any mapping $g\colon \mathfrak{X} \mapsto \mathfrak{X}$ satisfying

$$\gamma + \|g - f\|_C \leq \beta \tag{3.2}$$

there exists a trajectory $\mathbf{x} = \{x_n\} \in \mathbf{Tr}(g, \mathfrak{X}, 0)$ such that

$$|x_n - y_n| \leq \alpha(\gamma + \|g - f\|_C), \tag{3.3}$$

for all n for which $\mathbf{y}$ is defined.

A trajectory $\mathbf{x} = \{x_n\} \in \mathbf{Tr}(f, \mathfrak{Y}, 0)$ for some subset $\mathfrak{Y} \subseteq \mathfrak{X}$ is called a *cycle of period* N if $x_N = x_0$, while a finite pseudotrajectory $\mathbf{y} = \{y_n\}_{n=0}^N \in \mathbf{Tr}(f, \mathfrak{Y}, \gamma)$ is called a *γ-pseudocycle of period N* if $|y_N - y_0| \leq \gamma$ in addition to the inequalities (3.1). The dynamical system generated by the mapping $f\colon \mathfrak{X} \mapsto \mathfrak{X}$ is said to be *cyclically bi-shadowing with positive parameters α and β on a subset $\mathfrak{Y}$ of $\mathfrak{X}$* if for any given pseudocycle $\mathbf{y} \in \mathbf{Tr}(f, \mathfrak{Y}, \gamma)$ with $0 \leq \gamma \leq \beta$ and any mapping $g\colon \mathfrak{X} \mapsto \mathfrak{X}$ satisfying (3.2) there exists a proper cycle $\mathbf{x} \in \mathbf{Tr}(g, \mathfrak{X}, 0)$ of period N equal to that of $\mathbf{y}$ such that (3.3) holds for $n = 0, 1, \ldots, N$. Note that the cycle $\mathbf{x}$ here is required only to be in $\mathfrak{X}$ rather than in the subset $\mathfrak{Y}$ and that N need not be a minimal period.

The next theorem from [7] will be needed in the sequel.

Theorem 1. *Let $f \in \mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$, $f\colon \mathfrak{X} \mapsto \mathfrak{X}$, be* $\mathbf{s}$*-semihyperbolic on a compact subset $\mathfrak{Y}$ of $\mathfrak{X}$ with constants k, δ. Then it is both bi-shadowing and cyclically bi-shadowing on $\mathfrak{Y}$ with parameters*

$$\alpha(\mathbf{s}, k) = k\frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}, \tag{3.4}$$

$$\beta(\mathbf{s}, k, \delta) = \delta k^{-1} \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{\max\{\lambda_u - 1 + \mu_s, 1 - \lambda_s + \mu_u\}} \tag{3.5}$$

with respect to continuous mappings $g\colon \mathfrak{X} \mapsto \mathfrak{X}$.

The mapping f is said to be *ξ-expansive in $\mathfrak{X}$* if for any infinite trajectories $\mathbf{x}, \mathbf{y} \in \mathbf{Tr}(f, \mathfrak{X}, 0)$ either $\mathbf{x} = \mathbf{y}$ or $\sup_{-\infty < n < \infty}|x_n - y_n| \geq \xi$. A stronger version of the following theorem was proved in [6].

Theorem 2. *Let $f \in \mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$ be* $\mathbf{s}$*-semihyperbolic on a compact subset $\mathfrak{Y}$ of $\mathfrak{X}$ with constants k, δ. Then it is ξ-expansive on $\mathfrak{Y}$ with $\xi = \delta/k$.*

4. An Extremal Property of Topological Entropy

Let $f\colon \mathfrak{X} \mapsto \mathfrak{X}$ be a continuous mapping, where $\mathfrak{X}$ is an open bounded subset of $\mathbb{R}^d$ and let $\mathfrak{Z}$ be a compact subset of $\mathfrak{X}$. For a fixed positive integer N denote by $\mathbf{Tr}_{\pm N}(f, \mathfrak{Z})$ the totality of trajectories $\mathbf{x} = \{x_{-N}, \ldots, x_0, \ldots, x_N\}$ of f that are contained entirely in $\mathfrak{Z}$ and note that $\rho_N(\mathbf{x}, \tilde{\mathbf{x}}) = \sup_{-N \le n \le N} |x_n - \tilde{x}_n|$ is a metric on $\mathbf{Tr}_{\pm N}(f, \mathfrak{Z})$.

The *topological entropy* $\mathscr{E}(f, \mathfrak{Z})$ of f in $\mathfrak{Z}$ provides an index of how complicated the dynamics of f are in the set $\mathfrak{Z}$. There are various equivalent definitions [12] of $\mathscr{E}(f, \mathfrak{Z})$, and that used here is

$$\mathscr{E}(f, \mathfrak{Z}) = \lim_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{2N} C_\varepsilon(\mathbf{Tr}_{\pm N}(f, \mathfrak{Z})), \tag{4.1}$$

where $C_\varepsilon(\mathbf{Tr}_{\pm N}(f, \mathfrak{Z}))$ is the *ε-capacity* of the compact metric space $(\mathbf{Tr}_{\pm N}(f, \mathfrak{Z}), \rho_N)$, i.e, the binary logarithm of the maximal possible number of elements $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(p)}$ in $\mathbf{Tr}_{\pm N}(f, \mathfrak{Z})$ such that $\rho_N(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) \ge \varepsilon$ for all $i \ne j$. A rich theory of topological entropy has been developed for hyperbolic mappings (cf. [12] and the references therein) and many of the results remain valid for semihyperbolic mappings too. The following theorem is illustrative of such possible generalizations.

Theorem 3. *Let $f \in \mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$ be continuously* s*-semihyperbolic on a compact $\mathfrak{Z} \subset \mathfrak{X}$ with constants k, δ and $f(\mathfrak{Z}) = \mathfrak{Z}$. Then $\mathscr{E}(g, \overline{\mathscr{O}_{\delta/2k}(\mathfrak{Z})}) \ge \mathscr{E}(f, \mathfrak{Z})$ for each continuous mapping $g\colon \mathfrak{X} \mapsto \mathfrak{X}$ satisfying $\|g - f\|_C < \delta/(2k\alpha(\mathbf{s}, k))$, where $\alpha(\mathbf{s}, k)$ is defined by* (3.4).

Proof. A useful auxiliary concept in the proofs that follow is the *ε-entropy* of f, which is defined for $\varepsilon > 0$ as $\mathscr{E}_\varepsilon(f, \mathfrak{Z}) = \limsup_{N \to \infty}(1/2N) C_\varepsilon(\mathbf{Tr}_{\pm N}(f, \mathfrak{Z}))$. This is obviously nonincreasing in ε, so

$$\mathscr{E}(f, \mathfrak{Z}) = \lim_{\varepsilon \to 0} \mathscr{E}_\varepsilon(f, \mathfrak{Z}) = \sup_{\varepsilon > 0} \mathscr{E}_\varepsilon(f, \mathfrak{Z}). \tag{4.2}$$

Two additional essentially well known results [12] are required to complete the proof.

Lemma 2. *Let $f\colon \mathfrak{Z} \mapsto \mathfrak{Z}$ be a continuous ξ-expansive mapping. Then for any η and θ with $\eta \le \theta < \xi$ there exists a positive integer $N = N(\eta, \theta)$ such that $\rho_N(\mathbf{x}, \tilde{\mathbf{x}}) > \theta$ holds for all $\mathbf{x}, \tilde{\mathbf{x}} \in \mathbf{Tr}_{\pm N}(f, \mathfrak{Z})$ with $|x_0 - \tilde{x}_0| \ge \eta$.*

Proof. Suppose the contrary. Then for any positive integer N there exist trajectories $\mathbf{x}^{(N)}, \tilde{\mathbf{x}}^{(N)} \in \mathbf{Tr}_{\pm N}(f, \mathfrak{Z})$ satisfying $|x_0^{(N)} - \tilde{x}_0^{(N)}| \ge \eta$ and $\rho_N(\mathbf{x}^{(N)}, \tilde{\mathbf{x}}^{(N)}) \le \theta < \xi$. By compactness, without loss of generality, the sequences $\{\mathbf{x}^{(N)}\}$ and $\{\tilde{\mathbf{x}}^{(N)}\}$ can be assumed to converge componentwise to trajectories $\mathbf{x}^*$ and $\tilde{\mathbf{x}}^* \in \mathbf{Tr}_{\pm\infty}(f, \mathfrak{Z})$. Then $|x_0^* - \tilde{x}_0^*| \ge \eta$, but at the same time $|x_n^* - \tilde{x}_n^*| \le \theta < \xi$, $n = 0, \pm 1, \pm 2, \ldots$, which contradicts the ξ-expansivity of f. □

Lemma 3. *Let $f\colon \mathfrak{Z} \mapsto \mathfrak{Z}$ be a continuous ξ-expansive mapping with $f(\mathfrak{Z}) = \mathfrak{Z}$. Then $\mathscr{E}(f, \mathfrak{Z}) = \mathscr{E}_\theta(f, \mathfrak{Z})$ holds for every $\theta < \xi$.*

Proof. Fix θ with $0 < \theta < \xi$. Since $\mathscr{E}_\varepsilon(f, \mathfrak{Z})$ is nonincreasing in ε, then by (4.2) it suffices to prove that

$$\mathscr{E}_\eta(f, \mathfrak{Z}) \leq \mathscr{E}_\theta(f, \mathfrak{Z}) \tag{4.3}$$

for $\eta > 0$ sufficiently small. By Lemma 2 there exists an $N(\eta, \theta)$ such that $\mathbf{x}, \tilde{\mathbf{x}} \in \mathbf{Tr}_{\pm(N+N(\eta,\theta))}(f, \mathfrak{Z})$, $\rho_N(\mathbf{x}, \tilde{\mathbf{x}}) \geq \eta$ imply $\rho_{N+N(\eta,\theta)}(\mathbf{x}, \tilde{\mathbf{x}}) \geq \theta$ for any positive integer N. On the other hand, each trajectory $\mathbf{x} \in \mathbf{Tr}_{\pm N}(f, \mathfrak{Z})$ is the natural restriction of a corresponding trajectory in $\mathbf{Tr}_{\pm(N+N(\eta,\theta))}(f, \mathfrak{Z})$ because $f(\mathfrak{Z}) = \mathfrak{Z}$. Hence

$$C_\eta(\mathbf{Tr}_{\pm N}(f, \mathfrak{Z})) \leq C_\theta(\mathbf{Tr}_{\pm(N+N(\eta,\theta))}(f, \mathfrak{Z})),$$

from which immediately follows the desired inequality (4.3). □

Returning to the proof of Theorem 3, let f and g be mappings as in the statement of the theorem and fix θ for which $2\alpha(\mathbf{s}, k)\|g - f\|_C < \theta < \delta/k$, where $\alpha(\mathbf{s}, k)$ is defined by (3.4). Now f is ξ-expansive in $\mathfrak{Z}$ with $\xi = k^{-1}\delta$ by Theorem 2, so by Lemma 3, $\mathscr{E}(f, \mathfrak{Z}) = \mathscr{E}_\theta(f, \mathfrak{Z})$. In view of (4.2), to complete the proof of the theorem it remains to prove that $\mathscr{E}_\theta(f, \mathfrak{Z}) \leq \mathscr{E}_\sigma(g, \overline{\mathscr{O}_{\delta/2k}(\mathfrak{Z})})$, where $\sigma = \theta - 2\alpha(\mathbf{s}, k)\|g - f\|_C > 0$. This in turn will follow from

$$C_\theta(\mathbf{Tr}_{\pm N}(f, \mathfrak{Z})) \leq C_\sigma\left(\mathbf{Tr}_{\pm N}\left(g, \overline{\mathscr{O}_{\delta/2k}(\mathfrak{Z})}\right)\right) \tag{4.4}$$

for any positive N. To prove (4.4) let $\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(p)}\}$ denote the maximal subset of $\mathbf{Tr}_{\pm N}(f, \mathfrak{Z})$ satisfying $\rho_N(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) \geq \theta$, $i \neq j$. By Theorem 1 for each such $\mathbf{x}^{(i)}$ there exists a trajectory $\mathbf{y}^{(i)} \in \mathbf{Tr}_{\pm N}(g, \mathfrak{X})$ for which $\rho_N(\mathbf{y}^{(i)}, \mathbf{x}^{(i)}) \leq \alpha(\mathbf{s}, k)\|g - f\|_C < \delta/2k$. Hence $\mathbf{y}^{(i)} \in \mathbf{Tr}_{\pm N}(g, \mathscr{O}_{\delta/2k}(\mathfrak{Z}))$, $i = 1, \ldots, p$, and $\mathrm{P}_N(\mathbf{y}^{(i)}, \mathbf{y}^{(j)}) \geq \theta - 2\alpha(\mathbf{s}, k)\|g - f\|_C = \sigma$ for any $j \neq i$, from which (4.4) follows. This completes the proof of Theorem 3. □

5. Structural Stability Properties

Denote by $\mathscr{T}$ the totality of all sequences $\mathbf{x} + \{x_n\}_{n=-\infty}^{\infty}$ with $x_n \in \mathfrak{X}$, $n = 0, \pm 1, \pm 2, \ldots$. As usual (see [10]) we will consider $\mathscr{T}$ as a metric space with metric

$$\rho(\mathbf{x}, \tilde{\mathbf{x}}) = \sum_{n=-\infty}^{\infty} \frac{1}{4^{|n|}} |x_n - \tilde{x}_n|. \tag{5.1}$$

Since the set $\mathfrak{X}$ is assumed to be bounded, then the metric ρ is well defined. Let S denote the shift operator in $\mathscr{T}$, that is, $(\mathrm{S}\mathbf{x})_i = x_{i+1}$ for $i = 0, \pm 1, \pm 2 \ldots$, where $\mathbf{x} \in \mathscr{T}$. Let $\mathbf{Tr}_{\pm\infty}(g, \mathfrak{Y}) \subset \mathscr{T}$ be the totality of infinite trajectories $\mathbf{y} = \{y_n\} \subseteq \mathfrak{Y}$ of a mapping $g \in C(\mathfrak{X}, \mathbb{R}^d)$ belonging to the subset $\mathfrak{Y} \subseteq \mathfrak{X}$ and let $\mathfrak{Y}_1, \mathfrak{Y}_2$ be closed subsets of $\mathfrak{X}$ such that $g_1(\mathfrak{Y}_1) = \mathfrak{Y}_1$ and $g_2(\mathfrak{Y}_2) = \mathfrak{Y}_2$, where $g_1, g_2 \in C(\mathfrak{X}, \mathbb{R}^d)$. The restriction $g_1|_{\mathfrak{Y}_1}$ is said to be a *weak factorization of the restriction* $g_2|_{\mathfrak{Y}_2}$ if there exists a continuous (in the metric ρ) surjection Φ of the set $\mathbf{Tr}_{\pm\infty}(g_2, \mathfrak{Y}_2)$ onto the set $\mathbf{Tr}_{\pm\infty}(g_1, \mathfrak{Y}_1)$ which is shift invariant, i.e., with $\Phi \circ \mathrm{S} \equiv \mathrm{S} \circ \Phi$.

Another closely connected concept is that of weak conjugacy. The restricted mappings $g_1|_{\mathfrak{Y}_1}$ and $g_2|_{\mathfrak{Y}_2}$ are said to be *weakly conjugate* if there exists a continuous one-to-one correspondence Ψ between the set $\mathbf{Tr}_{\pm\infty}(g_1, \mathfrak{Y}_1)$ and $\mathbf{Tr}_{\pm\infty}(g_2, \mathfrak{Y}_2)$ which is shift invariant. By supposition, $\mathfrak{Y}_1, \mathfrak{Y}_2$ are closed subsets of the bounded set $\mathfrak{X}$. Hence they are compact, and then the metric spaces $\mathbf{Tr}_{\pm\infty}(g_1, \mathfrak{Y}_1)$ and $\mathbf{Tr}_{\pm\infty}(g_2, \mathfrak{Y}_2)$ are compact too. Therefore the mapping Ψ is homeomorphic. In other words, the restricted mappings $g_1|_{\mathfrak{Y}_1}$ and $g_2|_{\mathfrak{Y}_2}$ are weakly conjugate if the restrictions of the shift operator on the sets $\mathbf{Tr}_{\pm\infty}(g_1, \mathfrak{Y}_1)$ and $\mathbf{Tr}_{\pm\infty}(g_2, \mathfrak{Y}_2)$ are topologically conjugate.

The notion of weak factorization extends an analogue of persistence to semihyperbolic mappings. Weak conjugacy is a generalization of topological conjugacy of mappings and reduces to it in the case of invertible mappings [14]. The suitability of such generalizations in the analysis of noninvertible mappings is well known; see, for instance [14], Section 15.6. In particular, topological entropy is an invariant with respect to weak conjugacy and does not increase under weak factorization.

A point $x \in \mathfrak{Y}$ is called *$(\varepsilon, \mathfrak{Y})$-chain recurrent for f* if there exists an integer $L = L(\varepsilon)$ and points $x_0, x_1, \dots, x_L$ in $\mathfrak{Y}$ with $x = x_0 = x_L$ such that $|f(x_{i-1}) - x_i| < \varepsilon$ for $i = 1, 2, \dots, L$ [14]. Denote the totality of $(\varepsilon, \mathfrak{Y})$-chain recurrent points of f by $\mathrm{CR}(f, \varepsilon, \mathfrak{Y})$. The totality of *$\mathfrak{Y}$-chain recurrent points for f* is defined by $\mathrm{CR}(f, \mathfrak{Y}) = \bigcap_{\varepsilon > 0} \mathrm{CR}(f, \varepsilon, \mathfrak{Y})$. Clearly if $\overline{\mathfrak{Y}} \subseteq \mathfrak{X}$, then $\mathrm{CR}(f, \overline{\mathfrak{Y}})$ is compact and $f(\mathrm{CR}(f, \overline{\mathfrak{Y}})) = \mathrm{CR}(f, \overline{\mathfrak{Y}})$.

Lemma 4. *Let $\mathfrak{Y}$ be an open set with $\mathrm{CR}(f, \overline{\mathfrak{Y}}) \subset \mathfrak{Y} \subset \overline{\mathfrak{Y}} \subseteq \mathfrak{X}$ and let $f \in C(\mathfrak{X}, \mathbb{R}^d)$. Then there exists a nondecreasing function $q(\varepsilon, f)$ of $\varepsilon \geq 0$ with $q(\varepsilon, f) > 0$ for $\varepsilon > 0$ and $\lim_{\varepsilon \to 0} q(\varepsilon, f) = q(0, f) = 0$ such that $\mathrm{CR}(f, \varepsilon, \overline{\mathfrak{Y}}) \subseteq \mathscr{O}_{q(\varepsilon, f)}(\mathrm{CR}(f, \overline{\mathfrak{Y}}))$. In particular, there exists an $\varepsilon = \varepsilon(f, \mathfrak{Y}) > 0$ for which $\mathrm{CR}(f, \varepsilon, \overline{\mathfrak{Y}}) \subset \mathfrak{Y}$.*

Proof. Suppose the contrary. Then, for a certain $\varepsilon_0 > 0$, there exists a sequence $\mathbf{x}^{(k)} = \{x_n^{(k)}\}$ of $1/k$-pseudocycles of f with

$$\mathbf{x}^{(k)} \subseteq \overline{\mathfrak{Y}}, \qquad x_0^{(k)} \notin \mathscr{O}_{\varepsilon_0}(\mathrm{CR}(f, \overline{\mathfrak{Y}})). \tag{5.2}$$

Consider a limit point y of the sequence $\{x_0^{(k)}\}$ By the first inclusion (5.2) and by the definition of chain recurrence, $y \in \mathrm{CR}(f, \overline{\mathfrak{Y}})$, but by the second relation (5.2), $y \notin \mathscr{O}_{\varepsilon_0}(\mathrm{CR}(f, \overline{\mathfrak{Y}}))$. This contradiction proves the lemma. □

It is easy to estimate $q(\varepsilon, f)$ and $\varepsilon(f, \mathfrak{Y})$ numerically in Lemma 4.

Let $\mathfrak{Y}$ be an open set with $\overline{\mathfrak{Y}} \subseteq \mathfrak{X}$ and let $f \in \mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$ be continuously semihyperbolic on $\mathrm{CR}(f, \overline{\mathfrak{Y}})$ with $\mathrm{CR}(f, \overline{\mathfrak{Y}}) \subset \mathfrak{Y}$. By Lemma 1 there exists a $q_0 = q_0(f, \mathfrak{Y}) > 0$ such that f is semihyperbolic in $\mathscr{O}_{q_0}(\mathrm{CR}(f, \overline{\mathfrak{Y}}))$ with a certain $\delta, k, \mathbf{s}$. Denote by α and β the corresponding constants (3.4) and (3.5). By Lemma 1 there exist also $\gamma = \gamma(f, \mathfrak{X}) < \varepsilon(f, \mathfrak{Y})$ such that every g with $\|f - g\|_{\mathscr{L}} \leq \gamma$ is semihyperbolic in $\mathrm{CR}(f, \gamma, \overline{\mathfrak{Y}})$.

Theorem 4. *Let $\mathfrak{Y}$ be an open set with $\overline{\mathfrak{Y}} \subseteq \mathfrak{X}$ and let $f \in \mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$ be continuously $\mathbf{s}$-semihyperbolic in $\mathrm{CR}(f, \overline{\mathfrak{Y}}) \subset \mathfrak{Y}$ with some constants k, δ. If $g \in C(\mathfrak{X}, \mathbb{R}^d)$ with*

$\|g - f\|_C < \varepsilon$, where ε is such that

$$\varepsilon < \delta/(2k\alpha), \qquad q(\varepsilon, f) < q_0, \qquad \mathcal{O}_{q(\varepsilon)+\alpha\varepsilon}(\mathrm{CR}(f, \overline{\mathfrak{Y}})) \subset \mathfrak{Y}, \tag{5.3}$$

then the restricted mapping $f|_{\mathrm{CR}(f,\overline{\mathfrak{Y}})}$ is a weak factorization of $g|_{\mathrm{CR}(g,\overline{\mathfrak{Y}})}$.

Theorem 5. *Let $\mathfrak{Y}$ be an open set with $\overline{\mathfrak{Y}} \subseteq \mathfrak{X}$ and let $f \in \mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$ be continuously s-semihyperbolic in $\mathrm{CR}(f, \overline{\mathfrak{Y}}) \subset \mathfrak{Y}$ with some constants k, δ. If $g \in \mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$ with $\|g - f\|_{\mathscr{L}} < \gamma(f, \mathfrak{Y})$ then the restricted mapping $f|_{\mathrm{CR}(f,\overline{\mathfrak{Y}})}$ is weakly conjugate to $g|_{\mathrm{CR}(g,\overline{\mathfrak{Y}})}$.*

Theorem 4 is a weak form of persistence of semihyperbolic mappings, while Theorem 5 is a version of structural or topological stability for such mappings. The explicit estimates of the radii of persistence and structural stability here are useful in applications, such as in investigations of the effects of finite machine arithmetic on the computed behavior of chaotic mappings. Note that persistence is here a C^0-robust property, while structural stability is Lipschitz robust.

Proof of Theorem 4. Consider a mapping $g \in C(\mathfrak{X}, \mathbb{R}^d)$ such that $\|f - g\|_C < \varepsilon$, where ε satisfies (5.3). Denote $B(\mathbf{x}, \varepsilon) = \{\mathbf{y} \in \mathscr{T}\colon |x_i - y_i| \le \varepsilon,\ i = 0, \pm 1, \pm 2, \ldots\}$ and define the mapping Φ by

$$\Phi(\mathbf{x}) = B(\mathbf{x}, \alpha\varepsilon) \cap \mathbf{Tr}_{\pm\infty}(f, \mathrm{CR}(f, \overline{\mathfrak{Y}})), \qquad \mathbf{x} \in \mathbf{Tr}_{\pm\infty}(g, \mathrm{CR}(g, \overline{\mathfrak{Y}})),. \tag{5.4}$$

We need to prove that Φ is well defined, i.e., that the set

$$B(\mathbf{x}, \alpha\varepsilon) \cap \mathbf{Tr}_{\pm\infty}(f, \mathrm{CR}(f, \overline{\mathfrak{Y}})) \tag{5.5}$$

is nonempty and contains no more than one point.

We shall prove first that the set (5.5) is nonempty for each $\mathbf{x} \in \mathbf{Tr}_{\pm\infty}(g, \mathrm{CR}(g, \overline{\mathfrak{Y}}))$. To begin let us prove that for each positive integer m there exists a $1/m$-pseudocycle $\mathbf{x}^{(m)} = \{x_n^m\}_{n=0}^{N(m)} \subset \mathrm{CR}(g, 1/m, \overline{\mathfrak{Y}})$ of the mapping g, satisfying $|x_n^m - x_{n-m}| < 1/m$, $n = 0, 1, \ldots, 2m$. By the continuity of g there exists $\gamma_0 > 0$ such that each γ_0-pseudotrajectory $\mathbf{y} = y_0, y_1, \ldots, y_{2m}$ of g, with $y_0 = x_{-m}$, satisfies $\sup_{-m \le n \le m}|y_n - x_{n-m}| < 1/m$. That is, we can choose as $\mathbf{x}^{(m)}$ any γ_0-pseudocycle of g satisfying $|x_0^m - x_{-m}| < \gamma_0$ and $\mathbf{x}^{(m)} \subset \mathrm{CR}(g, 1/m, \overline{\mathfrak{Y}})$. Such pseudocycles exist as $x_{-m} \in \mathrm{CR}(f, \overline{\mathfrak{Y}})$ and so the existence of $\mathbf{x}^{(m)}$ is proved. Then each $\mathbf{x}^{(m)}$, $m = 1, 2, \ldots$ is a $(1/m + \|f - g\|_C)$-pseudocycle of f, i.e., $\mathbf{x}^{(m)} \subset \mathrm{CR}(f, 1/m + \|f - g\|_C, \overline{\mathfrak{Y}})$. Consequently, by Lemma 4, $\mathbf{x}^{(m)} \subset \mathcal{O}_{q(1/m + \|f-g\|_C, f)}(\mathrm{CR}(f, \overline{\mathfrak{Y}}))$ and so, by the inequality $q(\varepsilon, f) < q_0(f, \mathfrak{Y})$ of (5.3), $\mathbf{x}^{(m)} \subset \mathcal{O}_{q(\varepsilon, f)}(\mathrm{CR}(f, \overline{\mathfrak{Y}})) \subseteq \mathcal{O}_{q_0(f,\mathfrak{Y})}(\mathrm{CR}(f, \overline{\mathfrak{Y}}))$ for all sufficiently large m. Hence, in view of the definition of $q_0(f, \mathfrak{Y})$, the pseudocycles $\mathbf{x}^{(m)}$ belong to the region of semihyperbolicity of f for all sufficiently large m. From (3.4) and (3.5) it is seen that $\delta/\alpha \le \beta$. Then due to the first inequality of (5.3), $\varepsilon \le \beta/2k \le \beta$ holds. Therefore, by the cyclic bi-shadowing of Theorem 1, for each $\mathbf{x}^{(m)}$ with sufficiently large m there exists a cycle $\mathbf{y}^{(m)}$ of f with $|x_n^{(m)} - y_n^{(m)}| < \alpha\varepsilon$, $n = 0, 1, \ldots, 2m$. By the inclusion of (5.3), $\mathbf{y}^{(m)}$ belongs to $\mathfrak{Y}$ and, since the trajectory $\mathbf{y}^{(m)}$ is periodic, it then in fact belongs to the set $\mathbf{Tr}_{\pm\infty}(f, \mathrm{CR}(f, \overline{\mathfrak{Y}}))$. Therefore, by definition (5.1) of the metric ρ, any limit point of the m-shifted sequence $\mathrm{S}^{-m}\mathbf{y}^{(m)}$ belongs to the set (5.5). That is, the set (5.5) is nonempty for each $\mathbf{x} \in \mathbf{Tr}_{\pm\infty}(g, \mathrm{CR}(g, \overline{\mathfrak{Y}}))$. On the other hand, for any two trajectories $\mathbf{y}, \tilde{\mathbf{y}} \in B(\mathbf{x}, \alpha\varepsilon) \cap \mathbf{Tr}_{\pm\infty}(f, \mathrm{CR}(f, \overline{\mathfrak{Y}}))$, because of the first inequality of (5.3), the estimate $|y_i - \tilde{y}_i| < 2\alpha\varepsilon \leq \delta/k$ holds. By Theorem 2, the set (5.5) thus contains no more than one element, and so the mapping Φ is well defined.

We shall now prove that the mapping Φ is a surjection of the set $\mathbf{Tr}_{\pm\infty}(g, \mathrm{CR}(g, \overline{\mathfrak{Y}}))$ onto the set $\mathbf{Tr}_{\pm\infty}(f, \mathrm{CR}(f, \overline{\mathfrak{Y}}))$. In view of the definition of the mapping Φ we need only to construct for each $\mathbf{y} \in \mathbf{Tr}_{\pm\infty}(f, \mathrm{CR}(f, \overline{\mathfrak{Y}}))$ an element $\mathbf{x} \in \mathbf{Tr}_{\pm\infty}(g, \mathrm{CR}(g, \overline{\mathfrak{Y}}))$ with $|x_i - y_i| < \alpha\varepsilon$. As above, for each positive integer m there exist a $1/m$-pseudocycle $\mathbf{y}^{(m)}$ of the mapping f satisfying $|y_n^m - y_{n-m}| < 1/m$, $n = 0, 1, \dots, 2m$. As was mentioned above, due to the first inequality of (5.3) we have $\varepsilon \leq \beta$. So by the "cyclic part" of Theorem 1 for sufficiently large m there exist cycles $\mathbf{x}^{(m)}$ of g satisfying $|x_n^{(m)} - y_n^{(m)}| < \alpha\varepsilon$, $n = 0, 1, \dots, 2m$. It remains to define $\mathbf{x}$ as a limit point of the sequence $S^{-m}\mathbf{x}^{(m)}$ in the metric (5.1).

To prove the continuity of the mapping Φ, we shall suppose the contrary. Then there exist $\mathbf{x}, \mathbf{x}^{(m)} \in \mathbf{Tr}_{\pm\infty}(g, \mathrm{CR}(g, \overline{\mathfrak{Y}}))$, $m = 1, 2, \dots,$ such that

$$\rho(\mathbf{x}, \mathbf{x}^{(m)}) \to 0, \tag{5.6}$$

but $\rho(\Phi(\mathbf{x}), \Phi(\mathbf{x}^{(m)})) \geq \eta$ for some $\eta > 0$. In this case, without loss of generality, we may assume that

$$|(\Phi(\mathbf{x}))_0 - (\Phi(\mathbf{x}^{(m)}))_0| \geq \eta, \qquad m = 1, 2, \dots . \tag{5.7}$$

Choose a θ satisfying $2\alpha\varepsilon < \theta < \delta/k$; such θ exists by (5.3). Since by Theorem 2 the mapping g is ξ-expansive with $\xi = \delta/k$ on the set $\mathrm{CR}(f, \overline{\mathfrak{Y}})$, then by Lemma 2 there exist as a positive integer $N(\eta, \theta)$ not depending on m such that

$$\rho_{N(\eta,\theta)}(\Phi(\mathbf{x}), \Phi(\mathbf{x}^{(m)})) = \max_{-N(\eta,\theta) \leq i \leq N(\eta,\theta)} \{|y_i - y_i^{(m)}|\} \geq \theta \tag{5.8}$$

holds whenever (5.7) is valid. At the same time, from the definition of the mapping Φ it follows that $\rho_N(\Phi(\mathbf{x}), \mathbf{x}) \leq \alpha\varepsilon$ for any x and integer N. Hence

$$\begin{aligned}\rho_{N(\eta,\theta)}(\Phi(\mathbf{x}), \Phi(\mathbf{x}^{(m)})) &\leq \rho_{N(\eta,\theta)}(\Phi(\mathbf{x}), \mathbf{x}) + \rho_{N(\eta,\theta)}(\mathbf{x}, \mathbf{x}^{(m)}) + \rho_{N(\eta,\theta)}(\mathbf{x}^{(m)}, \Phi(\mathbf{x}^{(m)})) \\ &\leq 2\alpha\varepsilon + \rho_{N(\eta,\theta)}(\mathbf{x}, \mathbf{x}^{(m)}).\end{aligned}$$

Here $2\alpha\varepsilon < \theta$ by the definition of θ and $\rho_{N(\eta,\theta)}(\mathbf{x}, \mathbf{x}^{(m)}) \to 0$ in view of (5.6). Therefore, $\rho_{N(\eta,\theta)}(\Phi(\mathbf{x}), \Phi(\mathbf{x}^{(m)})) < \theta$ for sufficiently large m, which contradicts (5.8), and so the mapping Φ must be continuous.

The shift invariance identity $\Phi \circ \mathrm{S} \equiv \mathrm{S} \circ \Phi$ also follows from the definitions. Theorem 4 is thus completely proved. □

Proof of Theorem 5. Consider $g \in \mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$ satisfying $\|g - f\|_{\mathscr{L}} < \gamma(f, \mathfrak{Y})$. Introduce the family of mappings $g_\lambda = \lambda g + (1 - \lambda)f$, $0 \leq \lambda \leq 1$. Clearly,

$$\|g_\lambda - f\|_{\mathscr{L}} \leq \|g - f\|_{\mathscr{L}} < \gamma(f, \mathfrak{Y}). \tag{5.9}$$

Because the relation of weak conjugacy is transitive and the family $\{g_\lambda, 0 \le \lambda \le 1\}$ is a compact subset of $\mathscr{L}(\mathfrak{X}, \mathbb{R}^d)$, it remains to establish the following result:

Lemma 5. *There exists $\zeta > 0$ such that $g_\lambda|_{\mathrm{CR}(g_\lambda, \overline{\mathfrak{Y}})}$ is weakly conjugate to $g_{\lambda_*}|_{\mathrm{CR}(g_{\lambda_*}, \overline{\mathfrak{Y}})}$ for all $\lambda, \lambda_* \in [0,1]$ with $|\lambda - \lambda_*| < \zeta$.*

Proof. By (5.9) and the definition of $\gamma(f, \mathfrak{Y})$, for any $\lambda_* \in [0,1]$ the mapping $g_{\lambda_0^*}$ is continuously semihyperbolic in $\mathrm{CR}(f, \gamma(f, \mathfrak{Y}), \overline{\mathfrak{Y}})$. Since $\gamma(f, \mathfrak{Y}) < \varepsilon(f, \mathfrak{Y})$, then by Lemma 4, $\mathrm{Cr}(f, \gamma(f, \mathfrak{Y}), \overline{\mathfrak{Y}}) \subset \mathfrak{Y}$. Again by (5.9), any chain recurrent point of g_{λ_*} belonging to $\overline{\mathfrak{Y}}$ is a $\gamma(f, \mathfrak{Y})$-chain recurrent point of f. Hence

$$\mathrm{CR}\left(g_{\lambda_*}, \overline{\mathfrak{Y}}\right) \subseteq \mathrm{CR}(f, \gamma(f, \mathfrak{Y}), \overline{\mathfrak{Y}}) \subset \mathfrak{Y}.$$

Then by Lemmas 1 and 4 there exist $q_0 > 0$ and $\zeta_1 > 0$ such that all mappings $g_\lambda(x)$, $|\lambda - \lambda_*| < \zeta_1$, are uniformly semihyperbolic with some split s and constants k, δ in $\mathscr{O}_{q_0}(\mathrm{CR}(g_{\lambda_*}, \overline{\mathfrak{Y}}))$.

By $\mathrm{CR}(g_{\lambda_*}, \overline{\mathfrak{Y}}) \subset \mathfrak{Y}$ and Theorem 4, on the one hand, and the upper semicontinuity of the set $\mathrm{CR}(g_\lambda, \overline{\mathfrak{Y}})$ is in λ; on the other hand, there exists $\varepsilon > 0$ such that

$$\varepsilon < \delta/(2k\alpha) \tag{5.10}$$

and for each g_λ satisfying $\|g_\lambda - g_{\lambda_*}\|_C < \varepsilon$ the following properties are true:

(i) $g_{\lambda_*}|_{\mathrm{CR}(g_{\lambda_*}, \overline{\mathfrak{Y}})}$ is a weak factorization of $g_\lambda|_{\mathrm{CR}(g_\lambda, \overline{\mathfrak{Y}})}$ with the corresponding mapping

$$\Phi_\lambda(\mathbf{x}) = B(\mathbf{x}, \alpha\varepsilon) \cap \mathbf{Tr}_{\pm\infty}\left(g_{\lambda_*}, \mathrm{CR}\left(g_{\lambda_*}, \overline{\mathfrak{Y}}\right)\right), \qquad \mathbf{x} \in \mathbf{Tr}_{\pm\infty}(g, \mathrm{CR}(g, \overline{\mathfrak{Y}})).$$

(ii) $\mathrm{Cr}(g_\lambda, \overline{\mathfrak{Y}}) \subset O_{q_0}(\mathrm{CR}(g_{\lambda_*}, \overline{\mathfrak{Y}}))$.

Now, choose $\zeta > 0$ and $\lambda \in [0,1]$ such that $\zeta < \zeta_1$ and $|\lambda - \lambda_*| < \zeta$ imply that $\|g_\lambda - g_{\lambda_*}\| < \varepsilon$. On account of property (i), $g_{\lambda_*}|_{\mathrm{CR}(g_{\lambda_*}, \overline{\mathfrak{Y}})}$ is a weak factorization of $g_\lambda|_{\mathrm{CR}(g_\lambda, \overline{\mathfrak{Y}})}$. It remains to prove that Φ_λ is an injection and that the inverse mapping Φ_λ^{-1} is continuous.

To prove that Φ_λ is injective choose arbitrary $\mathbf{x}, \tilde{\mathbf{x}} \in \mathbf{Tr}_{\pm\infty}(g_\lambda, \mathrm{CR}(g_\lambda, \overline{\mathfrak{Y}}))$ such that $\mathbf{x} \neq \tilde{\mathbf{x}}$. By property (ii) and Theorem 2 the mapping g_λ is δ/k-expansive and so

$$\sup_{-\infty < i < \infty} |\mathbf{x}_i - \tilde{\mathbf{x}}_i| \ge \delta/k, \qquad \mathbf{x}, \tilde{\mathbf{x}} \in \mathbf{Tr}_{\pm\infty}(g_\lambda, \mathrm{CR}(g_\lambda, \overline{\mathfrak{Y}})), \qquad \mathbf{x} \neq \tilde{\mathbf{x}}. \tag{5.11}$$

On the other hand, by (5.10), property (i) and the inequality $|\lambda - \lambda_*| < \zeta$, the mapping Φ_λ satisfies

$$\sup_i |\Phi_\lambda(\mathbf{x})_i - x_i| < \alpha\varepsilon < \delta/2k \quad \text{and} \quad \sup_i |\Phi_\lambda(\tilde{\mathbf{x}})_i - \tilde{x}_i| < \alpha\varepsilon < \delta/2k. \tag{5.12}$$

Inequalities (5.11) and (5.12) imply

$$\sup_i |\Phi_\lambda(\mathbf{x})_i - \Phi_\lambda(\tilde{\mathbf{x}})_i| \geq \sup_i |\mathbf{x}_i - \tilde{\mathbf{x}}_i| - \sup_i |\Phi_\lambda(\mathbf{x})_i - \mathbf{x}_i| - \sup_i |\Phi_\lambda(\tilde{\mathbf{x}})_i - \tilde{\mathbf{x}}_i| > 0,$$

i.e., the mapping Φ_λ is an injection.

Finally, we need to prove that Φ_λ^{-1} is continuous in metric (5.1), for which we shall follow the proof of continuity of the mapping (5.4) above. Suppose the contrary. Then there exists a sequence $\mathbf{y}^{(m)} \in \mathbf{Tr}_{\pm\infty}(g_{\lambda_*}, \mathrm{CR}(g_{\lambda_*}, \overline{\mathfrak{Y}}))$, converging in the metric (5.1) to some $\mathbf{y} \in \mathbf{Tr}_{\pm\infty}(g_{\lambda_*}, \mathrm{CR}(g_{\lambda_*}, \overline{\mathfrak{Y}}))$ such that $\mathbf{x}^{(m)} = \Phi_\lambda^{-1}(\mathbf{y}^{(m)})$ does not converge to $\mathbf{x} = \Phi_\lambda^{-1}(\mathbf{y})$. Then without loss of generality we can suppose that $|x_0^{(m)} - x_0| \geq \eta$ for some positive η. Choose $\theta > 0$ with $2\alpha\varepsilon < \theta < \delta/k$; such a θ exists by (5.10). Then by Theorem 2 and Lemma 2 there exists a positive integer $N(\eta, \theta)$ satisfying

$$\rho_{N(\eta,\theta)}(\mathbf{x}^{(m)}, \mathbf{x}) = \max_{-N(\eta,\theta) \leq i \leq N(\eta,\theta)} |x_i^{(m)} - x_i| \geq \theta > 2\alpha\varepsilon.$$

From (5.11), (5.12) and the last inequality it follows that

$$\begin{aligned} \rho_{N(\eta,\theta)}(\mathbf{y}^{(m)}, \mathbf{y}) &\geq \rho_{N(\eta,\theta)}(\mathbf{x}^{(m)}, \mathbf{x}) - \rho_{N(\eta,\theta)}(\mathbf{y}^{(m)}, \mathbf{x}^{(m)}) - \rho_{N(\eta,\theta)}(\mathbf{y}, \mathbf{x}) \\ &\geq \theta - 2\alpha\varepsilon > 0. \end{aligned}$$

The inequality obtained contradicts $\lim_{m\to\infty} \rho(\mathbf{y}^{(m)}, \mathbf{y}) = 0$. Thus Φ_λ^{-1} must be continuous. The proofs of Lemma 5 and Theorem 5 are completed. □

6. Examples of Semihyperbolic Systems

Semihyperbolic mappings were introduced in Section 2 in a formal manner. Here we briefly describe some concrete examples. To begin we consider two degenerate cases of semihyperbolic systems. First, suppose that for all $x \in K$ the subspace E_x^u is empty and, concomitantly, $E_x^s = \mathbb{R}^d$. Then the class of semihyperbolic mappings is just that of the mappings which are locally contracting on some neighbourhood of K. On the other hand, if $E_x^s = \varnothing$ for all $x \in K$ and $E_x^u = \mathbb{R}^d$, then we have the class of all locally expanding mappings. This class is wider than the usual ones considered; see [9].

Semihyperbolic mappings also arise naturally as Lipschitz perturbations of hyperbolic mappings. We consider just a simple example. Let f be the canonical Smale horseshoe mapping on the square $Q = [-1,1] \times [-1,1]$ with compression factor $1/5$ and expansion factor 5 ([3], 3.5.1) and let $h(x)$ be a Lipschitz mapping on the square Q with $\mathrm{Lip}(h) < 2/3$. Then the mapping $H(x) = f(x) + h(x)$ is semihyperbolic on any compact $K \subset \mathrm{Int}(Q)$ with the natural splitting. If additionally $\|h\|_C < 1/5$, then clearly $\mathrm{CR}(H, Q) \subset \mathrm{Int}(Q)$. Note that H need not be invertible on $\mathrm{CR}(H,Q)$.

Another class of examples can be constructed as follows. Let I be a smooth immersion of the standard k-torus $\mathbf{T}^k$ into $\mathbb{R}^d$, with d sufficiently large for such an immersion to be possible, let F_A be an algebraic automorphism of $\mathbf{T}^k$ generated by an integer matrix A, and let π be the natural projection of a small neighbourhood $U \in \mathbb{R}^d$ of $I(\mathbf{T}^k)$ onto $I(\mathbf{T}^k)$. Then the mapping $x \mapsto f(x) = IF_A I^{-1}(\pi(x))$ is semihyperbolic in each compact subset of U. It must be emphasized that we do not suppose that $|\det(A)| = 1$. Note also that, even if $|\det(A)| = 1$, f is not hyperbolic on U because the projection is not invertible. Examples of this type arise naturally in the analysis of systems with a large parameter. For a detailed analysis of this example and perturbations of it, see [5], Section 5.

The final example deals with generalizations of ideas of Marotto [11, 15] on snap-back repellers. Let x_* be a hyperbolic fixed point of a smooth mapping f in $\mathbb{R}^d$. Denote by W^s and W^u, respectively, the local stable and unstable manifolds of f at the point x_*. Suppose that there is a point $x_0 \in W^u$ and a positive integer m with $f^m(x_0) = x_*$ and that the linear space $D_{x_0} T^u_{x_0}$ is transversal to $T^s_{x_*}$, where D_x is the derivative of f^m at the point x and T^u_z, T^s_x are the respective tangent spaces to the stable and unstable manifolds. This mapping is clearly not invertible. Let $E^s_x \oplus E^u_x$ be a continuous splitting such that $E^s_x = T^s_x$ for $x \in W^s$ and $E^u_x = T^u_x$ for $x \in W^u$. Since $x_0 \in W^u$, there exist points x_{-n}, $n > 0$, with $\lim_{n \to \infty} x_{-n} = x_*$ and $f(x_{-n}) = x_{-n+1}$. We may suppose that $|x_{-n+1}| > |x_{-n}|$ for $n > 0$. Let N be an arbitrary positive integer and write $r(N) = (|x_{-N}| + |x_{-N+1}|)/2$. For arbitrary $\varepsilon > 0$, denote $U(N, \varepsilon) = \mathscr{O}_{r(N)}(x_*) \cup \mathscr{O}_\varepsilon(x_0)$. For any positive integer M, define the mapping $F_{N,M,\varepsilon}$ on $U(N, \varepsilon)$ by

$$F_{N,M,\varepsilon} = \begin{cases} f(x), & \text{if } x \in \mathscr{O}_{r(N)}(x_*), \\ f^{N+m}(x), & \text{if } x \in \mathscr{O}_\varepsilon(x_{-N}). \end{cases}$$

If N is sufficiently large, then there exists a positive integer M such that for all sufficiently small $\varepsilon > 0$ this mapping is semihyperbolic on any compact subset of $U(N, \varepsilon)$ with the splitting $E^s_x \oplus E^u_x$. The calculations are straightforward, so will be omitted.

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References

[1] Anosov, D., Geodesic flows on closed Riemannian manifolds with negative curvature, *Proc. Steklov Inst. Math.* **90** (1967), 1–235.

[2] Aoki, N., and Hiraide, K., *Topological Theory of Dynamical Systems*, North-Holland, Dordrecht, 1994.

[3] Arrowsmith, D. K., and Place, C. M., *An Introduction to Dynamical Systems*, Cambridge University Press, Cambridge, 1990.

[4] Diamond, P., Kloeden, P., Kozyakin, V., Krasnosel'skii, M., and Pokrovskii, A., Robustness of trajectories of dynamical systems with hysteresis, *Dokl. Akad. Nauk.* **343** (1995), No. 1, 1–3 (English translation, *Phys. Dokl.*, to appear).

[5] Diamond, P., Kloeden, P., Kozyakin, V., and Pokrovskii, A., Computer robustness of semi-hyperbolic mappings, *Random & Comput. Dyn.*, to appear.
[6] Diamond, P., Kloeden, P., Kozyakin, V., and Pokrovskii, A., Expansivity of semihyperbolic Lipschitz mappings, *Bull. Austral. Math. Soc.* 51 (1995), No. 2, 301–308.
[7] Diamond, P., Kloeden, P., Kozyakin, V., and Pokrovskii, A., Robustness of observed behaviour of semi-hyperbolic dynamical systems (special issue dedicated to M. Krasnosel'skii's 75 anniversary) *Automat. Remote Control*, to appear.
[8] Krasnosel'skii, M. A., and Pokrovskii, A. V., *Systems with Hysteresis*, Nauka, Moscow, 1983 (English translation, Springer-Verlag, Berlin, 1992).
[9] Kifer, Y., *Random Perturbations of Dynamical Systems*, Birkhäuser, Boston, 1988.
[10] Mane, R., *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, Berlin, 1987.
[11] Marotto, F. R., Snack-back repellers imply chaos in $\mathbb{R}^n$, *J. Math. Anal. Appl.*, **63** (1978), 199–223.
[12] Martin, N., and England, J., *Mathematical Theory of Entropy*, Addison-Wesley, London, 1981.
[13] Ombach, J., Consequences of the pseudo-orbits tracing property and expansiveness, *J. Austr. Math. Soc. (Ser. A)*, **43** (1987), 301–313.
[14] Ruelle, D., *Elements of Differentiable Dynamics and Bifurcation Theory*, Academic, Boston, 1989.
[15] Shiraiwa, K., and Kurata, M., A generalization of a theorem of Marotto, *Nagoya Math. J.*, **82** (1981), 83–97.
[16] Walters, P., *On the Pseudo-Orbit Tracing Property and Its Relationship to Stability*, Springer-Verlag Lecture Notes in Mathematics, **668**, Springer-Verlag, New York, 1978, pp. 231–244.